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Transitive 7-Groups Degree 7^N (N=2)

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Abstract. In this paper we successfully classify transitive 7-groups of degree 49 and other unique properties of these groups.

Keywords: Abelian group, classification, degree, generator, isomorphism, p-groups, presentation, transitive.

1. Introduction

Let G be a group acting on a non-empty set Ω of size p^n. where p is an arbitrary fixed prime number (p= 7, n= 2). We therefore, determine up to isomorphism, the actual transitive p-groups of degree p^2 for p=7, and realize a total classification of transitive isomorphism with regards to degree. Again we will use GAP (Group Algorithms Programing) to establish the presentations and generators of the groups.

Lemma: 1.0.2

Let Ω be a set, and G be a group acting on Ω. Let H be a transitive subgroup of G on Ω. Then G is transitive on Ω.

Proof

Let $\alpha \epsilon \Omega$, since α^G is an orbit of G on Ω. It follows that $\alpha^G c \Omega$…(i)

Again since H is transitive on Ω, we must have $\alpha^H = \Omega$….(ii)

Claim: $\Omega c\ \alpha^G$

Let $\beta \epsilon \Omega$, then $\beta \epsilon \alpha^H$ using equation (ii), $\beta = \alpha^h$ for $h \epsilon H \leq G$. hence $\beta = \alpha^h$ for some $h \epsilon G$ and G is transitive on Ω and so $\Omega\ c\alpha$ and using (i) the results follows:

Lemma:1.0.3

Let $G \leq sym(\Omega)$ where Ω is a set. If $|G| < |\Omega|$. Then G is not transitive on Ω

Proof:

If G is transitive on Ω, then $|\alpha^G| = |\Omega| \forall \alpha \epsilon \Omega$, therefore, if

$|G|=|\alpha^G||G_\alpha|=|\Omega||\Omega| \forall \alpha \epsilon \Omega$, thus $|\Omega||G|$ and $|G| \geq |\Omega|$.

Lemma:1.0.4

Let G and M be finite polycyclic groups such that |G|=|M|. If G contains all the generators of M. then G=M, thus the groups G and M are just two different presentations of the same group.

Proof

Let $\{m_1,\ldots..m_s\}$ be the of all generators of M with $ml^{ni}=1$, where ni=0(mi) and (i=1, 2,…6). Let $\{g_1,g_{2\ldots}g_r\}$ be the set of generators of G with $g_i mi=1$. Suppose $x \epsilon M$, then we can always write $x=\prod_{i=1}^{s} mi = 1$, where $n_i \leq n_j$. since each $m_j \epsilon G$, and $m_j^{nj}=1$, we have m_j divide n_j and $m_j=\prod_{i=1}^{r} g_i^{mij}$ with $mij \leq mi$. therefore,

$X=\prod_{i=1}^{s} M_i^{ni} = \prod_{i=1}^{s} \square \prod_{j=1}^{r} (g_i^{mij})^{ni} = \prod_{ij=1}^{rs} g_i^{mijni}$ with $m_ijn_i < n_i$, consequently, M < G. M is a subgroup of G and since |G|=|M|, hence the result follows.

TRANSITIVE 7-GROUPS OF DEGREE 49 = 7^2

Let G be a transitive p-group of degree 49 = 7^2, acting on the set.

Ω = {1,2,3,4,5…49}. Then $G \leq sym(\Omega)$ as $|sym(\Omega)|$= 49! = 2.3.5.7.9.11

We must have $|G| = 7^n$ where n = 1,2,3 clearly, $n \neq 1$, n = 2 by lemma 1.0.3 above $|G| = 7^2$=49. G is certainly abelian and therefore,

$G \cong C_{49}$ or $G \cong C_7 \times C_7$. For transitivity, we have $\forall \alpha \epsilon \Omega$. $|G_{\alpha|\ =\ |G|,\ |\alpha}{}^G||G_\alpha| = 49$. Then we have the following possibilities.

$|\alpha^G|=1, |\alpha^G|= 49$……(i)

$|\alpha^G|=7, |\alpha^G|= 7$……..(ii)

$|\alpha^G|=49, |\alpha^G|= 1$…..(iii)

The equation (iii) holds for transitivity. Its well-known that there are exact 3 transitive 7-groups of degree 7^2 = 49 or order 49.

If $G \cong C_{49.}$ Then we may write.

G.= <a> where a^{49}= 1, $a \epsilon sym(49)$. Assume G is abelian. If $G \cong C49$.

take a= (1, 2, 3, 4 …..49)

If C7*C7, then $G \cong G_{2,,2}$ = < a, b: = 1, b^2 =1, ab = ba> with generators a= (1,2,3,4,5,6,7,) (8,9,10,11,12,13,14) (15,16,17,18,19,20) (22, 23,24,25,26,27,28) (29,30,31,32,33,34,35) (36,37,38,39,40,41,42) (43,44,45,46,47,48,49) b=(1,2,3,15,23,26,8) (2,16,18,22,49,27,25) (3,25,29,34,10,43,48) (4,17,24,28,36,11,44) (5,19,31,33,12,37,40) (6,21,32,38,13,41,46) (7,20,30,39,14,45,47).

Readily, $G_{1,3}$ and $G_{2,2}$ are transitive on Ω as shown above.

Lemma: 1.1

There are up to isomorphism, two (2) transitive 7-groups of degree 49 and order 49, namely the abelian groups $G_{1,2}$ and $G_{2,3}$ as shown above. When n=3, then |G| = 343 and for transitivity we must have |αG| = 49, $|G_\alpha|$ =7 ∀αεΩ.

Here G is non- abelian and we have the following possibilities for G,

$G \cong G_{1,3}$ = < a, b: a^{49}=1, b=1, ab =ba^4> with generators,

a= (1,2,3,4,5…..49)b=(1,10,13,21,33,39,27) (2,14,22,35,40,26,7 (2,8,15,23,29,36,41) (4,9,16,20,24,30,34) (5,11,17,25,37,42,31)(6,12,18,26,38,43,32).

Obtain by GAP- (see programme 1) the case $G_{2,3}$. We have the following presentation.

$G_{2,3}$ =< a, b, c : a^7 =1, b^7=1 ab=ba, c^7=1, ac= cab, bc =cb> with generators a= (1,9,16,23,30,38,7) (2,10,37,24,31,39,45) (3,11,18,25,32,40,46) (4,12,21,28,35,43,48) (8,15,22,29,36,44,49), b= (1,9,16,23,30,37,7) (2,17,24,38,45,31,10) (3,18,25,39,44,32,11) (4,19,26,40,46,12,33) (5,20,27,34,41,47,13) (6,21,35,42,46,28,14) (7,22,36,29,15,43,49). C= (1,7,19,41,31,,27,13) (2,8,20,32,26,12) (3,9,21,33,39,25,15) (4,10,22,34,40,25,16) (5,11,23,41,35,29,17) (6,12,24,13,6,42,18) (obtained by modification to programme 1) we can see that the above groups are transitive on Ω .

Lemma: 1.1.2

There are upto isomorphism two transitive 7-groups of degree 49 and order 343 namely the non-abelian groups $G_{1,3}$ and $G_{2,3}$ ab shown above

When n=4, |G|=2401 transitively, $|\alpha^G|$ = 49, $|G_\alpha|$ = 49,αεΩ, assuming G is non-abelian and the following are the possibilities of G.

$G \cong G_{1,4}$ = < $G_{1,3}$,c> where c^7 =1, $G_{1,3}$Δ $G_{1,4}$ or G≅ $G_{2,4}$= <$G_{2,3.}$ d> where d^7 = 1, $G_{2,3}$Δ $G_{2,4}$ for $G_{1,}$4 we have as a presentation.

$G_{1,4}$ = <a,b,c: a^{49}=1, b^7=1, ab= ba^4, c^7=1 ac=1 bc=cb> where a and b are same generators as those of $G_{1,3}$ and c= (1,5,4,8,16,11,23) for $G_{2,4}$ we have presentation:

$G_{2,4}$ = < a,b,c,d:a^7 =1, b^7=1, ab=ba, c^7=1, ac=cab, bc=cb, d^7=1, ad =dac, bd =db, cd =dc> we have the same generators of a,b,c as $G_{2,3}$ and we found that $G_{1,4} \cong G_{2,4.}$

Lemma 1:1.3

There are up to isomorphism 3 transitive 7-group of degree 49 and order 2401. Namely the non-abelian group $G_{1,4,}$ $G_{2.4,}$ $G_{3,4}$ as shown above. Here we therefore summarize our findings as shown on the table below:

Table 1: The number of transitive 7-groups of degree 49,isomorphism.

	$\|G\| = 7^n$	Number of transitive abelian 7-group of degree 49upto isomorphism	Number of transitive non-abelian 7-group of degree 49 upto isomorphism	Number of transitive 7-groups of degree 49 upto isomorphism
n=1	7	0	0	0
n=2	49	2	0	2
n=3	343	0	2	2
n=4 n=5 n=6	2401 16807 117649	0 0 0	3 3 3	3 3 3
Total		2	11	13

When n=5, then |G|= 16,807 and transitivity gave us $|\alpha^G|$=49, $|G_\alpha|$=343, therefore G will be non-abelian and have the following possibilities for G:

G=$G_{1,\ 5}$ = <$G_{1,\ 4}$, d> with d^7=1, $G_{1,\ 4}$ Δ $G_{1,\ 5}$ or G= $G_{2,5}$ =<$G_{2,4}$, e> with e^7=1, $G_{2,4}$ Δ $G_{2,5}$.

The presentation of $G_{1,5}$ is $G_{1,\ 5}$ =<a, b , c, d; a^{49}=1, b^7=1, ab =ba^6, c^7=1, ac=cab, bc=cb, d^7=1, ad=dbc, bd=db, cd=$da^4b^3c^2$; e^7=1, ae,=ea^2bcd^4, be= eb, ce = $eab^3c^2d^4$,de=ea^3bcd^4> with generators a,b,c,d, the same as $G_{1,4}$ and d=(1,7,8,11, 15,21,20) (3,12,13,10,12,24,2,)(4,16,24,28,32,40,4 8,)(obtained from programme 1.).

The group $G_{2,5}$ we have presentation: $G_{2,5}$=<a,b,c,d,e:a^7=1, b^7=1 ab=ba, c^7=1, ac=cab^3,bc=cb, d^7=1, ad dbc, bd=bd, cd=$da^4b^3c^2$, e^7=1, ae=ea^2bcd^4, be=eb, ce= $eab^3c^2d^4$,de =ea^2bcd^4> with Generators a, b, c, d give us the same presentation of $G_{2,4}$and:e=(1,20,13,16,11,9,24)(2,21,5,17,15,25,23) (3,10,8,19,13,7,26)(4,7.14.20,18,31,27) Therefore:

Lemma: 1.1.4 There is up to isomorphism 3 transitive 7-group of degree 49 and order 16807. The non-abelian group $G_{1,\ 5,}$ $G_{2,5}$ (of exponent 49) and $G_{3,5}$ (of exponent 7) as described above when n=6, |G|=117,649 and for transitivity $|\alpha^G|$=49, $|G_\alpha|$ 2401, αεΩ.

Here G is non-abelian and the following possibilities obtained.

G=$G_{1,6}$=<$G_{1,5}$,e> with e^7=1, $G_{1,5}$ Δ $G_{1,6}$ or G=$G_{2,6}$=<$G_{2,5}$, f> with f^7=1, $G_{2,5}$ Δ $G_{2,6}$

When $G_{1,6}$ we have a presentation: $G_{1,5}$=<a, b, c, d, e,: a^{49}=1,b^7=1, ab= ab^6, c^7=1,ac=cab, bc= cb, d^7=1,ad=dab^3c,bd =db, cd=dc, e^7=1, ae=eab^3cd^3, be=eb, ce =ec, de=ed> With generators a,b,c,d, have the same result as $G_{1,5}$ and e=(1,7,8,11,,15,21,20)(3,12,13,10,12,24,2) (4,16,24,28,32,40,48,) for $G_{2,4}$ we have the presentation: $G_{2,6}$ =<a, b, c, d, e: a^7=1,b^7=1, ab=ba, c^7=1, ac=cab^3, bc=cb, d^7=1, ad=dbc, bd=db, cd=$da^4b^3c^2$, e^7=1, ae= ea^2bcd^4,be=eb, ce=$eab^3c^2d^4$, de= ea^3bcd^4, f^7=1, af=$fa^4b^4c^2e$, bf=fb, cf=fa^3c^3e, df=$fabc^2d^2e^2$, ef=fe> with generators a, b, c, d, e, the same as $G_{2,5}$ and f=(1,16,21,9,23,5,20) (2,11,13,8,25,10,23) (3,15,7,26,24,19,17)

Lemma: 1.1.5 There is up to isomorphism three transitive 7-group of degree 49 and order 117649, that is the non-abelian group $G_{1, 6}$, $G_{2,6}$, $G_{3,6}$, of exponent 49 described above.

Proposition 1.1.6
There are up to isomorphism 13 transitive 7-groups of degree 7^2.,2 of these are abelian and the remaining 11 non-abelian, 6 are exponent 49 and 5 are of exponent 7.
Programme 1:

```
Gap>49: =symmetric Group (49)
Gap>a:=(1,2,3,,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,28,29,30,31,32,33,34,35,36,37,38,39,40,41,42,43,44,45,46,47,48,49).
Gap>b:=(1,6,11,16,21.26,31)(2,7,12,17,22,27)(3,8,13,18,23,28,32)(4,9,14,19,24,29,33)(5,10,15,20,25,30,34)
Gap>H: = subgroup (s49,[a,b])
Gap>centa:=(centralizer(s49,a );; centb:=centralizer (549,b)
Gap>x:=(1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,28,29,30,31,32,33,34,35,36,37,38,39,40,41,42,43,44,45,46,47,48,49).
Gap>y:= (1,2,3,4,5,6,7)(8,9,10,11,12,13,15,17)
Gap>k: =subgroup (s49,[x, y])
Gap>int: = intersection (k, centb)
Gap>diff: = Different(int,H)
Gap>req: = [ ]
Gap>for c in diff do
>if order (int,c) =7 then
>if order (int,c) < > 49 then
>if size (subgroup (s49,[a,b,c]) = then
>Add (req, c)
>fi
>fi
>fi
>od
Gap>req
Gap>size (req)
Gap>100
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